ON THE EFFICIENCY OF THE PERTURBED COLLOCATION TAU-METHOD FOR SOLVING FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

We investigate the accuracy of the perturbed collocation Tau-method for solving four points boundary value problem by varying \(N\) and observe that the accuracy of the numerical solutions obtained improves as \(N\) increases.

1. Introduction

In this paper, a fourth order non-linear differential equation has been solved using perturbed collocation Tau-method. We consider the non-linear differential equation of the form

\[
Q_4(x, y) \frac{d^4 y(x)}{dx^4} + Q_3(x, y) \frac{d^3 y(x)}{dx^3} + Q_2(x, y) \frac{d^2 y(x)}{dx^2} \\
+ Q_1(x, y) \frac{dy}{dx} + Q_0(x, y) = F(x)
\]

(1)

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with boundary conditions

\[
\begin{align*}
y(a) &= A \\
y(b) &= B \\
y^2(a) &= C \\
y^2(b) &= D
\end{align*}
\]

(2)

In order to solve equation (1, 2), we will use an approximate solution of the form:

\[
y_N(x) = \sum_{i=0}^{N} a_i x^i.
\]

(3)

On putting the approximation solution into slightly perturbed equation (1-2), we have

\[
Q_4(x, y) \frac{d^4 y_N(x)}{dx^4} + Q_3(x, y) \frac{d^3 y_N(x)}{dx^3} + Q_2(x, y) \frac{d^2 y_N(x)}{dx^2} \\
+ Q_1(x, y) \frac{dy_N(x)}{dx} + Q_0(x, y) y_N(x) = F(x) + H_N(x)
\]

(4)

with boundary conditions

\[
\begin{align*}
y_N(a) &= A \\
y_N(b) &= B \\
y_N^2(a) &= C \\
y_N^2(b) &= D
\end{align*}
\]

(5)

where \( H_N(x) \) is the Chebyshev Polynomial of degree \( N \) which takes the form

\[
H_N(x) = \tau_1 T_N(x) + \tau_2 T_{N-1}(x)
\]

(6)

and

\[
T_N(x) = \cos(N \arccos x), \quad a \leq x \leq b.
\]

(7)
2. Method of Linearization

In order to convert a non-linear equation to a linear one, Newton’s linearization scheme is employed. The Newton’s linearization scheme of order four defined by

\[ G = \frac{\Delta y \partial G}{\partial y} + \Delta y^{11} \partial G \partial y^{11} + \Delta y^{111} \partial G \partial y^{111} + \Delta y^{1111} \partial G \partial y^{1111} = 0 \]  

with

\[ \Delta y^k = y_{n+1}^k - y_n^k, \quad 1 \leq K \leq N \]

is used throughout this work.

Applying this in equation (1-2), the equation becomes a linear equation of the form:

\[ Q_4(x) \frac{d^4 y(x)}{dx^4} + Q_3(x) \frac{d^3 y(x)}{dx^3} + Q_2(x) \frac{d^2 y(x)}{dx^2} 
+ Q_1(x) \frac{dy(x)}{dx} + Q_0(x) = (x), \]

where \( H_n(x) \) is already defined in (6).

2.1. Non-linear problem and linearization method

Considering a non-linear perturbed fourth order boundary value problem defined by

\[ y^{iv} + (y^1)^2 + y(y^{111}) + 4x^2 - e^x (1 + x^2 - 4x) = 0 \]

with boundary conditions

\[ \begin{align*}
  y(a) &= A \\
  y(b) &= B \\
  y^1(a) &= C \\
  y^1(b) &= D
\end{align*} \]

such that \( a = 0, b = 1, A = 1, B = 1 + e, C = 1 \) and \( D = 2 + e. \)
This, from equation (8-9), we have

\[
\begin{align*}
\frac{\partial G}{\partial y} &= y^{111} \\
\frac{\partial G}{\partial y^T} &= -2y^1 \\
\frac{\partial G}{\partial y^{11}} &= 0 \\
\frac{\partial G}{\partial y^{111}} &= y \\
\frac{\partial G}{\partial y^{1w}} &= 1
\end{align*}
\]

(13)

Putting (13) in (8-9), we have

\[
\Delta y_n(y_n^{111}) - \Delta y_N^1(2y_N^1) + \Delta y^{111}(y_n) + \Delta y^{1w} = -4x^2 - e^x(1 + x^2 - 4x),
\]

(14)

where

\[
\Delta y^k = y^k_{n+1} - y^k_n, \quad 1 \leq k \leq N
\]

using (9) in (14), we have

\[
(y_{n+1} - y_n)y^{111} - (y^1 - y^1)\Delta y^1(2y^1) + y^{111} - y^{111}(y_n)
\]

\[
+ (y^{1w} - y^w) = -4x^2 + e^x(1 + x^2 - 4x).
\]

(15)

Thus, by opening the bracket and compare with equation (11), the linearized equation becomes:

\[
(y_{N,n+1}(x) + y_{N,n}(x))y^{111} - 2y_{N,n}(x)y^{111}_{N,n+1}(x) + y^{111}(x)y_{N,n+1}(x))
\]

\[
= -(y_{N,n}(x))^2 - y_{N,n}(x)y^{111}_{N,n}(x) - 4x^2 + e^x(1 + x^2 - 4x).
\]

(16)

3. Numerical Example

We consider the fourth order non-linear differential equation:

\[
y^{1w} - (y^{111})^2 + y(y^{11} - 1) - 2e^x \sin(x) - 1 = 0
\]

(17)
with the boundary conditions

\begin{align}
  y(0) &= 2 \\
  y'(0) &= 1 \\
  y^{(n/2)} &= e^{\Pi/2} \\
  y^{(1/2)}(x) &= e^{\Pi/2} - 1
\end{align} \tag{18}

The initial guess is

\[ Y_{N,0}(x) = e^x - \sin(x). \] \tag{19}

In order to solve this problem, Newton linearization scheme is used.

Using equations (8) and (9), we have

\begin{align}
  \frac{\partial G}{\partial y} &= y^{11} - 1 \\
  \frac{\partial G}{\partial y^1} &= 0 \\
  \frac{\partial G}{\partial y^{11}} &= y \\
  \frac{\partial G}{\partial y^{111}} &= -2y^{111} \\
  \frac{\partial G}{\partial y^{1111}} &= -1 \tag{20}
\end{align}

substituting (20) into (8) and making use of (9), we have

\[ (y_{N,n+1}(x) - 2y_{N,n+1}(x)y_{N,111}^{N,n+1} + y_{N,n}(x)y_{N,111}^{N,n+1}(x) + (y_{N,111}^{N,n} - 1)y_{N,n+1}(x) \]

\[ = y_{N,n}(x)y_{N,111}(x) - (y_{N,111}^{N,n})^2 + 2(1 + 2e^x \sin(x)) \] \tag{21}

with boundary conditions

\begin{align}
  y_{N,n+1}(0) &= 2 \\
  y_{N,n+1}'(0) &= 1 \\
  y_{N,n+1}(\Pi/2) &= e^{\Pi/2} \\
  y_{N,n+1}'(\Pi/2) &= e^{\Pi/2} - 1 \tag{22}
\end{align}

\[ y_{N,0}(x) = e^x - \sin(x). \] \tag{23}
Equation (21) is the linearized equation using Newton’s linearization scheme.

In order to solve (21), an approximation solution of the form (3) is assumed.

Thus, substituting equation (3) into a slightly perturbed equation (21), we have

\[
y^{(0)}_{N,n+1}(x) - 2y_{N,x}(x)y^{(1)}_{N,n+1} + y_{N,n}(x)y^{(1)}_{N,n+1}(x) \\
+ (y^{(1)}_{N,n} - 1)y_{N,n+1}(x) - H_N(x) \\
= y_{N,n}(x)y^{(1)}_{N,n}(x) - (y^{(1)}_{N,n}(x))^2 + 2(1 + 2e^x \sin(x)),
\]

where

\[
H_N(x) = \tau_1 T_N(x) + \tau_2 T_{N-1}(x),
\]

\[
T_n(x) = \cos(N^* \arccos(x)), \quad 0 \leq x \leq \Pi/2.
\]

We now solve equation (24) with the boundary condition (22) for various values of \(N\).

For case \(N = 4\), our approximation solution (3) becomes

\[
y_{4,n+1}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.
\]

Substituting (27) into the perturbed problem (24) and making use of the initial guess (23), we have

\[
(e^x + \sin(x) - 1)a_0 + (xe^x + x \sin(x) - x)a_1 \\
+ (x^2 e^x - 2 \sin(x) + x^2 \sin(x) + 2e^x - x^2)a_2 \\
+ (x^3 e^x + x^3 \sin(x) - 12 \cos(x) - 12e^x + 6xe^x - x^3)a_3 \\
+ (x^4 e^x + x^4 \sin(x) - 12x^2 \sin(x) - 48 \cos(x) \\
+ 12x^2 e^x - 48xe^x - 24 - x)a_4 - T_1 T_4(x) - T_2 T_3(x) \\
= 1 + 2(2 \sin(x) - \cos(x))e^x.
\]
Collocate equation (28) at point
\[ x = x_1. \] (29)

The above equations together with (29) form seven equations with seven unknowns. The unknowns are \( a_0, a_1, a_2, a_3, a_4, T_1, T_2. \)

If the equations are written in the form of matrix \( Ax = B \) and solved, then we have
\[
\begin{align*}
a_0 &= 2, \quad a_1 = 1, \quad a_2 = 0.3646184, \quad a_3 = 0.012487, \quad a_4 = -0.0020280584, \\
T_1 &= 0.01102488, \quad T_2 = 0.1998359.
\end{align*}
\]

For case, \( N = 5 \), our approximate solution (3) becomes
\[ y_{5,n+1}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5. \] (30)

Substituting (30) into (24) making use of (22) and solving the resulting equation, we have
\[
\begin{align*}
a_0 &= 2, \quad a_1 = 1, \quad a_2 = 0.5392581, \quad a_3 = 0.064902, \quad a_4 = -0.1796324, \\
a_5 &= 0.07901274, \quad T_1 = -0.531587, \quad T_2 = -0.194093.
\end{align*}
\]

<table>
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<tr>
<th>( X )</th>
<th>Approximate Solution</th>
<th>Exact Solution</th>
<th>Error</th>
</tr>
</thead>
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<td>2.000000</td>
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Table 2. Error estimate for case $N = 5$

<table>
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4. Conclusion

We used perturbed collocation Tau-method to solve non-linear four points boundary value problems by varying $N$ and observed that the accuracy of the numerical solutions obtained improves as $N$ increases. This method requires solving large set of matrix equations and this requires more computational efforts for high accuracy.

References