Further result on the strong summability of Fourier series

K. RAUF‡*, J. O. OMOLEHIN† and D. J. EVANS‡

†Department of Mathematics, University of Ilorin, Ilorin, Kwara State, Nigeria
‡Faculty of Engineering and Computing, Nottingham Trent University, Burton Street, Nottingham NG1 4BU, UK

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This article deals with some special cases which are extension of the strong summability of Fourier series with constant factor. We obtain a new equivalent form of inequalities

\[ A \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta \leq \int_0^{2\pi} \left\{ \int_0^1 (1 - \rho) |\phi'(z)|^2 \, d\rho \right\}^{r/2} \, d\theta \leq B \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta, \]

\[ \int_0^{2\pi} \left\{ \int_0^1 (1 - \rho)^{q-1} |\phi'(z)|^q \, d\rho \right\}^{r/q} \, d\theta \leq C \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta, \]

\[ D \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta \leq \int_0^{2\pi} \left\{ \int_0^1 (1 - \rho)^{p-1} |\phi'(z)|^p \, d\rho \right\}^{r/p} \, d\theta. \]

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1. Introduction

The theory of Fourier series and integrals was known to have developed from the study of the boundedness of integral operators in \( L^p \) spaces by Fourier and his contemporaries. It plays an important role and has become an indispensable tool in the analysis of certain periodic phenomena which are studied in natural sciences and engineering.

During the last decades, many researchers like Susan [1], Sunouchi [2], Littlewood and Paley [3], De Guzman [4], Iantchenko and Sjöstrand [5], Iantchenko [6], Rosenblum and Rovnyak [7], Mossahed [8], and a lot of others have studied the idea and extended it to

*Corresponding author. Email: balk_r@yahoo.com
some problems like vibrations, planetary motion and wave motion. The object of much of the research has been to find sufficient conditions to be satisfied by function $\phi$ in order that its Fourier series may converge, either throughout the interval or at a particular point.

In this article, we are interested in formulating further conditions on function $\phi$, which will guarantee the validity of the following theorem.

**Theorem 1.1** Let $f(\theta)$ be a function of the class $L^r$, then

$$A \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta \leq \int_0^{2\pi} \left\{ \int_0^1 (1 - \rho) |\phi'(z)|^2 \, d\rho \right\}^{r/2} \, d\theta$$

$$\leq B \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta,$$

$$\int_0^{2\pi} \left\{ \int_0^1 (1 - \rho)^{q-1} |\phi'(z)|^q \, d\rho \right\}^{r/q} \, d\theta \leq C \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta,$$

$$D \int_0^{2\pi} |\phi(e^{i\theta})|^r \, d\theta \leq \int_0^{2\pi} \left\{ \int_0^1 (1 - \rho)^{p-1} |\phi'(z)|^p \, d\rho \right\}^{r/p} \, d\theta. \quad (3)$$

Inequality (1) is due to Littlewood and Paley [3], whereas equations (2) and (3) are due to Sunouchi [9].

More precisely, we look for necessary and sufficient conditions on the non-negative function $\phi$ and parameter $p, q$ and $r$ satisfying $r > 1, 1 < p \leq q < \infty$.

Throughout this article, we let $f(\theta) \sim \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$ be an integrable function with period $2\pi$ and $\phi(z) = \sum (a_n + ib_n)z^n$ for $z = pe^{i\theta}$, then $f(0)$ becomes the boundary function of the harmonic function $R\Phi(z)$.

**Theorem 1.2** Let $f(\theta) \in L^p$ and suppose

$$u(\theta) \leq \left( \int_0^1 (1 - \rho)^q |\phi'(z)|^{q+1} \, d\rho \right)^{1/(q+1)}, \quad v(\theta) \leq \left( \int_0^1 (1 + \rho)^q |f'(z)|^{q+1} \, d\rho \right)^{1/(q+1)}$$

and

$$g(\theta) \leq \left( \int_0^1 ((1 - \rho)|\phi'(z)|^2 \, d\rho \right)^{1/2}, \quad h(\theta) \leq \left( \int_0^1 ((1 + \rho)|f'(z)|^2 \, d\rho \right)^{1/2}$$

satisfying

$$\Phi(\theta) = \max(|\phi(z)|; z \in S(\theta)), \quad F(\theta) = \max(|f(z)|; z \in S(\theta)),$$

with

$$|\phi'| \leq \frac{C\Phi}{(1 - \rho)}, \quad |f'| \leq \frac{C\Phi}{(1 + \rho)},$$

where $S(\theta)$ is a kite-shaped region, $C = C(p, q, r)$ is a constant independent of $\phi$ and $f$, $k = k(\phi, f)$ and $(1/p) + (1/q) = 1$ with $r > 1, 1 < p \leq q < \infty$. 
Then,

\[
A \int_0^{2\pi} (|\phi(e^{i\theta})||f(e^{i\theta})|)^p d\theta \leq \int_0^{2\pi} \left\{ \int_0^{1} ((1 - \rho^2)|\phi'(z)||f'(z)|)^2 d\rho \right\}^{p/2} d\theta, \tag{4}
\]

\[
\int_0^{2\pi} \left\{ \int_0^{1} (1 - \rho^2)^q |\phi'(z)||f'(z)|)^{q+1} d\rho \right\}^{p/(q+1)} d\theta \leq C \int_0^{2\pi} (|\phi(e^{i\theta})||f(e^{i\theta})|)^p d\theta, \tag{5}
\]

\[
D \int_0^{2\pi} (|\phi(e^{i\theta})||f(e^{i\theta})|)^p d\theta \leq \int_0^{2\pi} \left\{ \int_0^{1} ((1 - \rho^2)|\phi'(z)||f'(z)|)^{r+1} d\rho \right\}^{p/(r+1)} d\theta. \tag{6}
\]

Inequality (4) shows that the left-hand side of equation (6) is not less than the right-hand side of equation (5).

**Proof**

Let

\[
I = \left\{ \int_0^{2\pi} \left( \int_0^{1} (1 - \rho^2)^q |\phi'(z)||f'(z)|)^{q+1} d\rho \right)^{p/(q+1)} d\theta \right\}^{1/p}
\]

then

\[
I \leq \left\{ \int_0^{2\pi} \left[ \left( \int_0^{1} (1 - \rho^2)^q |\phi'(z)|^{q+1} d\rho \right)^{p/(q+1)} \right] d\theta \right\}^{1/p}
\]

\[
\leq \left\{ \int_0^{2\pi} \left[ \left( \int_0^{1} (1 - \rho^2)^q |\phi'(z)|^{2} (1 - \rho)^{q-2} |\phi'(z)|^{q-1} d\rho \right)^{p/(q+1)} \right] d\theta \right\}
\]

\[
\leq \left\{ \int_0^{2\pi} \left[ (g(\theta)^2(C_1 \Phi)^q)^{p/(q+1)} \right] (h(\theta)^2(C_2 F)^q)^{p/(q+1)} d\theta \right\}^{1/p}
\]

\[
\leq C^{2((q-2)/(q+1))} \left\{ \left( \int_0^{2\pi} g(\theta)^2 p/(q+1) \Phi p/(q-2)/(q+1) d\theta \right) \right\}
\]

\[
\times \left\{ \left( \int_0^{2\pi} h(\theta)^2 p/(q+1) F p/(q-2)/(q+1) d\theta \right) \right\}^{1/p}
\]

\[
\leq C^{2((q-2)/(q+1))} \left\{ \left( \int_0^{2\pi} |g(\theta)|^{p} d\theta \right)^{1/p} \right\}^{2/(q+1)} \left[ \left( \int_0^{2\pi} |\phi|^p d\theta \right)^{1/p} \right]^{(q-2)/(q+1)}
\]

\[
\times \left\{ \left( \int_0^{2\pi} |h(\theta)|^{p} d\theta \right)^{1/p} \right\}^{2/(q+1)} \left[ \left( \int_0^{2\pi} F p d\theta \right)^{1/p} \right]^{(q-2)/(q+1)}
\]

Summability of Fourier series
Thus, we have

\[ \leq C^{2((q-2)/(q+1))} \left\{ \left( \int_0^{2\pi} |g(\theta)|^p \, d\theta \right)^{1/p} \right\}^{2/(q+1)} \left\{ \left( \int_0^{2\pi} |\phi|^p \, d\theta \right)^{1/p} \right\}^{(q-2)/(q+1)} \]

\[ \times \left\{ \left( \int_0^{2\pi} |h(\theta)|^p \, d\theta \right)^{1/p} \right\}^{2/(q+1)} \left\{ \left( \int_0^{2\pi} |f|^p \, d\theta \right)^{1/p} \right\}^{(q-2)/(q+1)} \]

by Holder's inequality

\[ \leq C^{2(q-2)/(q+1)} \left[ \left( \int_0^{2\pi} |\phi|^p \, d\theta \right)^{1/p} \right] \left[ \int_0^{2\pi} |f|^p \, d\theta \right]^{1/p} \]

\[ \leq \frac{k}{p} C^{2(q-2)/(q+1)} \left[ \int_0^{2\pi} (||f||)^p \, d\theta \right]^{1/p} \quad \text{[see ref. 10, Theorem 230, p. 166].} \]

In addition, we have,

\[ \left\{ \int_0^{2\pi} \left[ \left( \int_0^1 (1 - \rho^2) |\phi'(z)||f'(z)|^2 \right)^{p/2} \, d\rho \right]^{1/p} \right\} \]

\[ \leq \left\{ \int_0^{2\pi} \left[ \left( \int_0^1 (1 - \rho^2) |\phi'(z)||f'(z)|^{q+1} (C_1 \Phi)^{1-q} \, d\rho \right) \right. \]

\[ \times \left. \left( \int_0^1 (1 + \rho)^q |f'(z)|^{q+1} (C_2 F)^{1-q} \, d\rho \right) \right]^{p/2} \, d\theta \right\}^{1/p} \]

\[ \leq C^{1-q} \left\{ \int_0^{2\pi} (|u(\theta)|^{q+1} \Phi^{1-q})(|v(\theta)|^{q+1} F^{1-q}) \, d\theta \right\}^{1/p} \]

\[ \leq C \left\{ \int_0^{2\pi} (|u(\theta)|^{p(q+1)/2} \Phi^{p(1-q)/2} |v(\theta)|^{p(q+1)/2} F^{p(1-q)/2}) \, d\theta \right\}^{1/p} \]

\[ \leq C^{1-q} \left[ \left( \int_0^{2\pi} |u(\theta)|^p \, d\theta \right)^{1/p} \right]^{(q+1)/2} \left[ \left( \int_0^{2\pi} \Phi^p \, d\theta \right)^{1/p} \right]^{(1-q)/2} \]

\[ \times \left[ \left( \int_0^{2\pi} |v(\theta)|^p \, d\theta \right)^{1/p} \right]^{(q+1)/2} \left[ \left( \int_0^{2\pi} F^p \, d\theta \right)^{1/p} \right]^{(1-q)/2} \]

\[ \leq C^{1-q} \left[ \int_0^{2\pi} |u(\theta)|^p \, d\theta \right]^{1/p} \left[ \int_0^{2\pi} |v(\theta)|^p \, d\theta \right]^{1/p} \quad \text{(by Holder's inequality)} \]

\[ \leq \frac{k}{p} C^{1-p} \left[ \int_0^{2\pi} (|u||v|)^p \, d\theta \right]^{1/p} . \]

Thus, we have

\[ \left[ \int_0^{2\pi} (|g||h|)^p \, d\theta \right]^{1/p} \leq C^{((1-q^2) - 2q + 4)/(q+1)} \left[ \int_0^{2\pi} (|u||v|)^p \, d\theta \right]^{1/p} . \]

Hence the proof of the theorem.
Remark \ If we put \( p = q = 1 \), \(|g||h| = \phi'\) and \(|u||v| = \phi\), then we obtain a useful result due to Sunouchi, Littlewood and Paley.

2. Some results and lemmas

In this section, we state the main result to be proved in this article.

**Theorem 2.1** Suppose \( \{S_n(e^{i\theta})\} \) and \( \{\tau_n(e^{i\theta})\} \) denote the sequence of partial sums and the arithmetic mean of Fourier power series generated by function \( \phi(e^{i\theta}) \in H^p \), respectively. Then

\[
A \int_0^{2\pi} |\phi(e^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} \left( \sum_{n=1}^{\infty} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{2/n} \right)^{p/2} \, d\theta, \tag{7}
\]

\[
\int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{n} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{q+1} \right)^{p/(q+1)} \, d\theta \leq C \int_0^{2\pi} |\phi(e^{i\theta})|^p \, d\theta, \tag{8}
\]

\[
D \int_0^{2\pi} |\phi(e^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{n} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{r+1} \right)^{p/(r+1)} \, d\theta, \tag{9}
\]

with \( p, q \) and \( r \) defined as in Theorem 1.2. We need to show that

\[
\int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{n} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{q+1} \right)^{p/(q+1)} \, d\theta \leq \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{n} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{r+1} \right)^{p/(r+1)} \, d\theta \tag{10}
\]

To prove Theorem 2.1, we need the following lemma.

**Lemma 2.1** Let \( f_1, f_2, \ldots \) be a sequence of integrable functions and let \( S_{n,v} \) denote the \( v \)th partial sum of the Fourier series of \( f_n \). Where \( r > 1 \) and \( m > 1 \). The inequality is also valid for conjugate series.

**Proof** The proof is immediate from ref. [11] if we set \( m = m + 1 \). \( \blacksquare \)
Lemma 2.2 Let \( S_{n,v}(\rho, \theta) \) denote the sum derived from \( f_n(\theta) \) similarly as \( S_v(\rho, \theta) \) from \( f(\theta) \). Then under the hypothesis of Lemma 2.1, we have

\[
\int_0^{2\pi} \left( \sum_{n=1}^{\infty} |S_{n,k_n}(\rho_n, \theta)|^{m+1} \right)^{2/(m+1)} d\theta \leq A \int_0^{2\pi} \left( \sum_{n=1}^{\infty} |f_n(\theta)|^{m+1} \right)^{r/(m+1)} d\theta,
\]

where \( 0 \leq \rho_n \leq 1 \), with

\[
S_v(\rho, \theta) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \rho^n, \quad f(\theta) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \rho^n.
\]

The sums \( S_{n,v} \) may be replaced by the conjugate ones.

Proof This is immediate from Lemma 2.1 by Zygmund’s argument by a slight modification. ■

Lemma 2.3 Let \( 0 \leq \rho_n \leq 1 \) and \( \Delta_n \) denote an arbitrary interval situated \( (\rho_n, 1) \). Then under the hypothesis of Lemma 2.1, we have

\[
\int_0^{2\pi} \left( \sum_{n=1}^{\infty} |S_{n,k_n}(\rho_n, \theta)|^{m+1} \right)^{r/(m+1)} d\theta \leq A \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{\Delta_n} \int_{\Delta_n} |f_n(\rho, \theta)|^{m+1} d\rho \right)^{r/(m+1)} d\theta.
\]

The sums \( S_{n,v} \) may be replaced by the conjugate ones.

Proof This follows from Lemma 2.2. ■

Proof of Theorem 2.1 Suppose \( \rho_n = 1 - (1/(n+1)) \) and \( \Delta_n = (\rho_n, \rho_{n+1}) \).

Let

\[
I = \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{n} |S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^{q+1} \right)^{p/(q+1)} d\theta.
\]

Then,

\[
I \leq C \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{|S'_n(\rho_n e^{i\theta})|^{q+1}}{n^{q+2} \rho_n^{(q+1)n}} + \sum_{n=1}^{\infty} \frac{1 - \rho_n}{n^{q+2} \rho_n^{n}} \sum_{v=0}^{n-1} \rho_n^{-v-1} \frac{|S'_v(\rho_n e^{i\theta})|^{q+1}}{n^{q+1}} \right)^{p/(q+1)} d\theta.
\]

\[
\leq C \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{|S'_n(\rho_n e^{i\theta})|^{q+1}}{n^{q+2} \Delta_n} \int_{\Delta_n} |\phi'(\rho e^{i\theta})|^{q+1} d\rho + \sum_{n=1}^{\infty} \frac{1}{n^{q+3}} \right)^{p/(q+1)} \times \sum_{v=0}^{n-1} \frac{1}{\Delta_n} \int_{\Delta_n} |\phi'(\rho e^{i\theta})|^{q+1} d\rho \right)^{p/(q+1)} d\theta \quad \text{(by Abel Transformation)}
\]

\[
\leq C \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{n^{q+2}}{n^{q+2}} \int_{\Delta_n} (1 - \rho)^q |\phi'(\rho e^{i\theta})|^{q+1} d\rho \right)^{p/(q+1)} d\theta
\]

\[
\leq C \int_0^{2\pi} \left( \int_0^1 (1 - \rho)^q |\phi'(\rho e^{i\theta})|^{q+1} d\rho \right)^{p/(q+1)} d\theta
\]

\[
\leq C \int_0^{2\pi} |\phi(e^{i\theta})|^p d\theta \quad \text{[by equation (8)]}
\]
Similarly, let \( \rho_n = 1 - (1/n) \). Then,

\[
|\phi'(e^{i\theta})| = \sum_{v=1}^{\infty} \nu_{cv} \rho^{v-1} e^{i\theta} \\
= \sum_{v=1}^{\infty} S'_v \rho^{v-1} - \rho \sum_{v=1}^{\infty} S'_v \rho^{v-1} \\
= (1 - \rho) \sum_{v=1}^{\infty} S'_v \rho^{v-1} .
\]

Therefore,

\[
\int_0^1 (1 - \rho)^r |\phi'|^{r+1} d\rho \\
= \sum_{n=1}^{\infty} \int_{\rho_n}^{\rho_{n+1}} (1 - \rho)^r |\phi'|^{r+1} d\rho \\
\leq \sum_{n=0}^{\infty} \frac{1}{n^{r+2}} \left[ (1 - \rho_n) \sum_{n=1}^{\infty} |S'_v| \rho^{v-1} \right]^{r+1} \\
\leq C \left[ \sum_{v=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{n=1}^{\infty} |S'_v| \rho_{n+1}^{v-1} \right)^{r+1} \right] \\
\leq C \left[ \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=1}^{\infty} |S'_v| \rho_{n+1}^{v-1} \right)^{1/p} \left( \sum_{v=n+1}^{\infty} \nu^{(v+1)(r+1)/p} \right)^{(r+1)/p} \right] \\
+ \left[ \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=n+1}^{\infty} \nu^{(v+1)(r+1)/p} \right)^{1/p} \left( \sum_{v=n+1}^{\infty} |S'_v|^p \rho_{n+1}^{p(v-1)} \right)^{(r+1)/p'} \right] \\
\leq C \left[ \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=1}^{\infty} |S'_v|^p \right)^{1/p} n^{(r+1)/p'} \right] + \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=n+1}^{\infty} |S'_v|^p \nu^{(v+1)(r+1)/p} \right)^{1/p'} \\
\times \left( \sum_{v=n+1}^{\infty} \nu^{(v+1)(r+1)/p} \right)^{(r+1)/p'} \\
\leq C \left[ \sum_{n=1}^{\infty} \frac{1}{n^{(r+1)/p - p(r+2)+1}} \left( \sum_{v=1}^{n} |S'_v|^p \right) + \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=n+1}^{\infty} |S'_v|^p \right) \nu^{(v+1)(r+1)/p} \right].
\]
\[ \leq C \left[ \sum_{n=1}^{\infty} \frac{1}{n^{(r+1)/p} - p(r+2)} \left( \sum_{v=1}^{\infty} \frac{|S_v^{(r+1)}|}{v^{(r+2)}} \right) + \sum_{n=1}^{\infty} \frac{1}{n^{2r+3}} \left( \sum_{v=n+1}^{\infty} \frac{|S_v^{(r+1)}|}{v^{(r+3)}} \right) \right] \]

\[ \leq C \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n^{(r+1)/p} - p(r+2)} + \frac{1}{n^{2r+3}} \right) \left( \sum_{v=n+1}^{\infty} \frac{|S_v^{(r+1)}|}{v^{(r+3)}} \right) \right] \]

\[ \leq CN \sum_{v=n+1}^{\infty} \frac{|S_v^{(r+1)}|}{v^{r+3}} \]

with \(|S_v^{(r+1)}/(w+1) = |S_v - \tau_v|\). Then the result follows.

3. Applications of our main result

The following theorems are based on strong summability theorem concerning Lacunary sequences of partial sums [see ref. 12].

**Theorem 3.1** Suppose \( f(\theta) \in H^p (p > 1) \), \( \{P_n\} \) and \( \{d_n\} \) be arbitrary non-decreasing sequences of positive integers such that,

\[ \frac{P_{n+1}}{d_{n+1}} = O(P_{n+2} - P_{n+1}). \]

Then,

\[ \int_0^{2\pi} \left( \frac{|S_{pv} - \tau_{pv}|^{q+1}}{q+1} \right)^{p/(q+1)} d\theta \leq C \int_0^{2\pi} |f(\theta)|^p d\theta \quad \text{for all } q \geq 2. \quad (11) \]

**Proof** Let

\[ \left( 1 - \frac{1}{P_{n+1}}, 1 - \frac{1}{P_{n+2}} \right), \quad \text{if } P_{n+2} < 2P_{n+1}, \]

\[ \left( 1 - \frac{1}{P_{n+1}}, 1 - \frac{1}{2P_{n}} \right), \quad \text{if } P_{n+2} \geq 2P_{n+1}, \]

and

\[ \rho_{n+1} = \frac{1}{(P_{n+1} + 1)}. \]

Then, the result follows from equation (8).

**Theorem 3.2** Let \( \{P_n\} \) and \( \{d_n\} \) be defined as in Theorem 3.1 such that

\[ \frac{P_{n+1}}{n+1} = O(P_{n+2} - P_{n+1}). \]

Then

\[ \sum_{\gamma=1}^{\infty} |S_{\gamma}|^{(m+1)/\gamma} < \infty \quad (m \geq 0). \quad (12) \]

**Proof** It is immediate from equation (10) by Kronecker’s theorem.
Theorem 3.3 Suppose $f(\theta) \in H^p(p > 1)$, then the sequence $\{1, 2, \ldots\}$ can be divided into two complementary sequences $\{(n + 1)_l\}$ and $\{(m + 1)_k\}$, depending on $\theta$, such that $S_P^{(\theta)}(n + 1)_k \to f(\theta)$, the series $\sum (1/d_{m+1})$ converges, where $\sum (1/d_{n+1}) = \infty$ and $P_{n+1}/d_{n+1} = O(P_{n+2} - P_{n+1})$.

Proof This follows from the theorem due to Zygmund [13].

4. Numerical investigation

In this section, we present the results obtained by the relation between the strong summability factor $d(x)$ and the lacunary sequence $P(x; n)$ described by

$$
\frac{1}{d(x)} \sim \frac{P'(x; n)}{P(x; n)} = F(x; n).
$$

The results are tabulated in the following table.

If $d(x) = 1$ and $P(x; n) = n^\xi$ for $n = 2, 3, \ldots, k$ then

<table>
<thead>
<tr>
<th>$P(x; n)$</th>
<th>$2^\xi$</th>
<th>$3^\xi$</th>
<th>$4^\xi$</th>
<th>$5^\xi$</th>
<th>$6^\xi$</th>
<th>$7^\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(x; n)$</td>
<td>0.69314718</td>
<td>1.09861229</td>
<td>1.38629436</td>
<td>1.60943791</td>
<td>1.79175947</td>
<td>1.94591015</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P(x; n)$</th>
<th>$8^\xi$</th>
<th>$9^\xi$</th>
<th>$10^\xi$</th>
<th>$11^\xi$</th>
<th>$12^\xi$</th>
<th>$13^\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(x; n)$</td>
<td>2.07944154</td>
<td>2.19722458</td>
<td>2.30258509</td>
<td>2.39789527</td>
<td>2.48490665</td>
<td>2.56494936</td>
</tr>
</tbody>
</table>

From the table mentioned earlier, we observed that when $n = 2$ we obtained the result due to Kolmogoraff [14]. Other results could be obtained from the relation.

References
